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A survey of allocation models in search theory.

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A SURVEY OF ALLOCATION MODELS
IN
SEARCH THEORY

by

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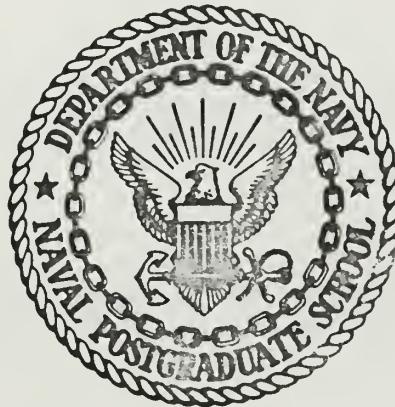
THESIS

A Survey of Allocation Models
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Search Theory

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Brian David Engler

September 1970

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Search Theory

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ABSTRACT

The problem of optimally allocating available efforts to search for an object at sea comprises a major class of problems in naval warfare. This thesis presents in some detail Koopman's classic two-region and continuous search models, along with the n-region discrete model which provides some continuity between the two. Brief summaries of four of the more important extensions to the basic theory are also included.

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I. INTRODUCTION

A major class of problems in naval warfare consists of those involving the search for an object at sea, the position of which is unknown but is distributed in accordance with a known law of probability. The question of interest to this thesis is how to allocate available resources (or effort) in a search of this type, as for an enemy submarine or a downed pilot.

Early research was done by B. O. Koopman who, in 1946, wrote up the results of studies performed by what would later be known as the Operations Evaluation Group (OEG) of the United States Navy [5]. His models continue to be the basis for further studies into search theory, and it is, therefore, natural to begin any paper on allocation of searching effort with a derivation of Koopman's original model.

II. KOOPMAN'S TWO-REGION MODEL

Koopman began the study of search effort distribution with the simplest case: consider two areas in either one of which an object may be located with a certain probability. The initial assumption to be made in this situation is that the random search model will be used to describe the search, regardless of the actual searching pattern used. This is a conservative yet realistic assumption since, practically speaking, any search, regardless how systematic, has a certain amount of randomness inherent in it due to navigational and other errors. The results obtained using this model are fairly simple and require no further assumptions concerning the particular detection law except for the observer's sweep width.

A review of the development of the random search model is an essential basis for a full understanding of Koopman's development.

A. DEVELOPMENT OF THE RANDOM SEARCH MODEL

It is assumed that the stationary target is equally as likely to be in any one location within the given search region of area A as in any other (i.e., the target is uniformly distributed in A). It is assumed, further, that the searcher has no set plan for observation.

Several definitions are necessary at this point:

$\overline{P}(X)$. . . the lateral range for a certain observer and target in the existing environmental conditions. It is a graphical representation of the cumulative probability of detecting a single target passing at some lateral range, X, from a particular detection device under given environmental conditions.

L. . . the path length of the observer in A, which is divided into N equal segments each approximating a straight line.

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L/N. . .   the length of each of the N path segments. By the
           random search assumption, each is independent.

W . . .   the sweep width of the observer, equal to the area
           under the lateral range curve. $\left(\int_{-R_m}^{R_m} \overline{P}(x)\,dx \right)$

Rm. . .   the maximum lateral range from the observer at which
           the target is detectable.

Let: event A be the event that the target is in area A,

           event B be the event that the target is detectable
           along any segment L/N (i.e., that it lies within the
           area L/Nx2Rm)

           event C be the event that the target is detected
           along the segment L/N. Then $\overline{C}$ is the event that it
           is not detected along L/N.

           event D be the event that the target is detected.
           Then $\overline{D}$ is the event of non-detection.

It is to be understood that event A is a condition throughout
the following:

$$P(B) = \frac{2R_m \cdot L/N}{A}$$

$$P(C\,|\,B) = E[\overline{P}(x)] = \frac{1}{2R_m} \int_{-R_m}^{R_m} \overline{P}(x)\,dx = \frac{W}{2R_m}$$

Thus,

$$P(C) = P(C\,|\,B) \cdot P(B) = \frac{WL}{NA}$$

$$P(\overline{C}) = 1 - \frac{WL}{NA}$$

And,

$$P(D) = \left(1 - \frac{WL}{NA}\right)^N = e^{N\,\ln\left(1 - \frac{WL}{NA}\right)}$$

When $\left(\frac{WL}{NA}\right)$ is small, which is most often the case:

$$\ln\left(1 - \frac{WL}{NA}\right) \approx -\frac{WL}{NA}$$

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So,

$$P(\bar{D}) = e^{N\left(-\frac{WL}{NA}\right)}$$

and finally,

$$P(D|A) = 1 - e^{-\frac{WL}{A}} \qquad \text{[Equation 1]}$$

At this point, the condition on event A has been inserted for a reason which will soon be evident.

One final assumption which should be noted is that $2Rm \ll L/N$, thus permitting the overlap of segments to be disregarded .

The expression $\frac{WL}{A}$ is called the coverage factor, and measures the amount of effort expended in searching the area A.

It should be noted that by the very nature of the observer's random path selection, the probability of detecting a target within a certain area, A, given that it is in area A, will approach, but never quite equal, one. In other words, the observer can never be certain of entirely covering A with the 2Rm-wide swath of coverage he is cutting out along his search path L, due to the possibility of crossing an area already searched.

B. DERIVATION OF THE TWO-REGION MODEL

In Koopman's simplest model it is known that:

(1) A target is located within one of two regions of areas A_1 and A_2 with probabilities p_1 and p_2 respectively, where: $p_1 + p_2 = 1$, $p_1 > 0$, $p_2 > 0$.

(2) The target is stationary, which implies p_1 and p_2 are constant.

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(3) The target has a uniform distribution in whichever region it is located.

(4) L, which is the observer's total path or track length is such that: $L=L_1 +L_2$, where L_1 and L_2 are the track lengths in A_1 and A_2.

A good measure of effort available is the length of track along which the observer can search. Since a maximum L exists and is known, the problem to which this model should provide an answer can be stated as: What is the best, or optimal, distribution of L between A_1 and A_2 such that the chance of detecting the target is as large as possible?

Using the events A and D defined previously, it is a fact that:

$$P(D) = P(A) \cdot P(D|A)$$

In the two region model:

$$p_1 = P(\text{target is in } A_1)$$
$$p_2 = P(\text{target is in } A_2)$$

and the random search equation becomes:

$$P(D|A_1) = 1 - e^{-\frac{WL_1}{A_1}}$$
$$P(D|A_2) = 1 - e^{-\frac{WL_2}{A_2}}$$

Thus, the mathematical statement of the problem is as follows:

maximize $\quad P(D) = p_1 \left(1 - e^{-\frac{WL_1}{A_1}}\right) + p_2 \left(1 - e^{-\frac{WL_2}{A_2}}\right)$

subject to: $\quad L_1 + L_2 = L$

$$L_1 \geq 0 \quad , \quad L_2 \geq 0$$

The objective function may be simplified:

$$P(D) = p_1 - p_1 e^{-\frac{WL_1}{A_1}} + p_2 - p_2 e^{-\frac{WL_2}{A_2}}$$
$$= p_1 + p_2 - p_1 e^{-\frac{WL_1}{A_1}} - p_2 e^{-\frac{WL_2}{A_2}}$$

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Thus

$$P(D) = 1 - p_1 e^{-\frac{WL_1}{A_1}} - p_2 e^{-\frac{WL_2}{A_2}}$$ [Equation 2]

Letting $L_1 = x$ and $L_2 = L - L_1 = L - x$, for simplicity, the maximization of [Equation 2] can be converted to an equivalent minimization problem, namely:

$$\underset{x}{\text{minimize}} \quad f(x) = p_1 e^{-\frac{Wx}{A_1}} + p_2 e^{-\frac{W(L-x)}{A_2}}$$

subject to: $\quad 0 \leq x \leq L$

This problem may be solved both graphically and analytically, the latter of which will be followed in this thesis, though using a method somewhat different from Koopman's, since graphs soon become useless when the model is extended beyond two regions. Koopman's graphical solution is useful, however, as an aid to understanding just what needs to be done in optimizing the allocation of search track as a measure of searching effort in this simple case, and may be found in reference [5].

In order to solve this minimization problem using the Lagrangian approach, the inequality constraint must first be gotten rid of. This can be accomplished by letting $x = u^2$ and adding a slack variable, so that the problem becomes:

$$\underset{u}{\text{minimize}} \quad f(u^2) = p_1 e^{-\frac{Wu^2}{A_1}} + p_2 e^{-\frac{W(L-u^2)}{A_2}}$$

subject to: $\quad u^2 + s^2 = L$

The Lagrangian function is:

$$L(\lambda, u, s) = f(u^2) + \lambda(L - u^2 - s^2)$$

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For which the following necessary conditions must hold at a stationary point:

(1) $L_\lambda = \dfrac{\partial L}{\partial \lambda} = L - u^2 - s^2 = 0$

(2) $L_s = \dfrac{\partial L}{\partial s} = -2 s \lambda = 0 \Rightarrow s\lambda = 0$

(3) $L_u = \dfrac{\partial L}{\partial u} = 2u\,f'(u^2) - 2u\lambda$
$= 2u\left(f'(u^2) - \lambda\right) = 0$

From (2), if $s \neq 0$, then $\lambda = 0$, and from (3), $uf'(u^2)=0$. This in turn implies that either $u=0$ ($x=0$), or $f'(u^2)=f'(x)=0$.

Again from (2), it might be that $s=0$. Then from (1), $u^2=x=L$. Hence, there are three cases to be examined:

Minimum at   (a) $x=0 \Rightarrow f'(x=0) \geq 0$
(b) $x=L \Rightarrow f'(x=L) \leq 0$
(c) $x \ni f'(x)=0$

These cases may be visualized most readily be referring to Koopman's graphical approach in [5].

Since, $f(x) = P_1 e^{-\frac{Wx}{A_1}} + P_2 e^{-\frac{W(L-x)}{A_2}}$,

$f'(x) = -\dfrac{W}{A_1} P_1 e^{-\frac{Wx}{A_1}} + \dfrac{W}{A_2} P_2 e^{-\frac{W(L-x)}{A_2}}$ is obtained.

Case (a):   if $\exists$ a minimum at $x=0 \Rightarrow f'(0) \geq 0$ (the curve depicting $f(x)$ can slope upward or be level for any $x \in [0, L]$ as x goes from 0 to L).

Now, $f'(0) = -\dfrac{W}{A_1} P_1 e^{-\frac{W \cdot 0}{A_1}} + \dfrac{W}{A_2} P_2 e^{-\frac{W(L-0)}{A_2}}$

$= -\dfrac{W}{A_1} P_1 + \dfrac{W}{A_2} P_2 e^{-\frac{WL}{A_2}} \geq 0$

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or,

$$\frac{\cancel{W}}{A_2} \, P_2 e^{\frac{-WL}{A_2}} \geqslant \frac{\cancel{W}}{A_1} \, P_1$$

Thus,

$$\frac{P_2}{A_2} \, e^{-\frac{WL}{A_2}} \geqslant \frac{P_1}{A_1} \qquad \text{[Equation 3]}$$

Case (b): if $\exists$ a minimum at $x=L \Rightarrow f'(L) \leqslant 0$.

By a similar exercise:

$$\frac{P_1}{A_1} \, e^{-\frac{WL}{A_1}} \geqslant \frac{P_2}{A_2} \qquad \text{[Equation 4]}$$

Case (c): if $\exists$ a minimum at $x \ni f'(x)=0$, this value of x must be found.

Let

$$f'(x) = -\frac{W}{A_1} \, P_1 e^{-\frac{Wx}{A_1}} + \frac{W}{A_2} \, P_2 e^{-\frac{W(L-x)}{A_2}} = 0$$

In this equation

$$e^{-\frac{W(L-x)}{A_2}} = e^{-\frac{WL}{A_2}} \cdot e^{\frac{Wx}{A_2}}$$

By transposing,

$$\frac{W P_2}{A_2} \, e^{-\frac{WL}{A_2}} \cdot e^{\frac{Wx}{A_2}} = \frac{W P_1}{A_1} \, e^{-\frac{Wx}{A_1}}$$

or,

$$\frac{P_2}{A_2} \, e^{-\frac{WL}{A_2}} = \frac{P_1}{A_1} \, e^{-\frac{Wx}{A_1}} \cdot e^{-\frac{Wx}{A_2}}$$

Note that,

$$e^{-\frac{Wx}{A_1}} \cdot e^{-\frac{Wx}{A_2}} = e^{-Wx\left(\frac{1}{A_1}+\frac{1}{A_2}\right)} = e^{-Wx\left(\frac{A_1+A_2}{A_1 A_2}\right)}$$

Taking the natural log of both sides,

$$\ln \frac{P_2}{A_2} + \frac{WL}{A_2} = \ln \frac{P_1}{A_1} + Wx \left(\frac{A_1 + A_2}{A_1 A_2} \right)$$

Multiplying both sides by $\frac{1}{W}$ and transposing yields:

$$x \left(\frac{A_1 + A_2}{A_1 A_2} \right) = \frac{L}{A_2} + \frac{1}{W} \ln \frac{P_1}{A_1} - \frac{1}{W} \ln \frac{P_2}{A_2}$$

In this equation,

$$\frac{1}{W} \left[\ln \frac{P_1}{A_1} - \ln \frac{P_2}{A_2} \right] = \frac{1}{W} \left(\frac{P_1}{P_2} \cdot \frac{A_2}{A_1} \right)$$

Therefore, in the case where $f'(x)=0$, the minimum value of $f(x)$ is at:

$$x = \left(\frac{A_1 A_2}{A_1 + A_2} \right) \left(\frac{L}{A_2} + \frac{1}{W} \ln \frac{P_1}{P_2} \frac{A_2}{A_1} \right)$$

or,

$$x = \frac{1}{A_1 + A_2} \left(A_1 L + \frac{A_1 A_2}{W} \ln \frac{P_1}{P_2} \frac{A_2}{A_1} \right) \qquad \text{[Equation 5]}$$

It is obvious from the above equations that there exists one point, x, in the interval [0, L] which minimizes $f(x)$ and in turn maximizes $P(D)$ subject to the constraints on track length.

Perhaps more important, however, note that [Equation 3] and [Equation 4] set down threshold values for $\frac{P_1}{A_1}$ and $\frac{P_2}{A_2}$, which, following Koopman's example, will be defined as the probability densities for the two regions, and labeled ρ_1 and ρ_2, respectively. These values, then, are such that unless ρ_1 exceeds the value $\rho_2 e^{-\frac{WL}{A_2}}$, all searching effort should be concentrated in A_2, and likewise, if $\rho_2 \leqslant \rho_1 e^{-\frac{WL}{A_1}}$, A_2 should be ignored and all effort sent to A_1.

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At this point, a final simplification of notation is necessary
in order to clarify the preceding argument. Note that a better
measure of searching effort than merely the observer's track length,
L_1, in area A_1, is the area swept within that region, or WL_1. Thus
the expressions:

$$\emptyset_1 = \frac{WL_1}{A_1} \qquad \text{and} \qquad \emptyset_2 = \frac{WL_2}{A_2}$$

are defined by Koopman as the density of search effort in the two
regions. Finally, the total searching effort will be defined as
follows for the two-region case:

$$\overline{\Phi} = A_1 \emptyset_1 + A_2 \emptyset_2 = W(L_1 + L_2) = WL$$

where $\emptyset_1$, $\emptyset_2 \geqslant 0$, and the total area is $A = A_1 + A_2$.

In summary, then, for the two-region search model, if:

$$\rho_1 \leqslant \rho_2 e^{-\emptyset_2} \qquad \qquad \text{[Equation 3']}$$

then all searching effort should be confined to A_2. $(\text{i.e. } L_2 = L)$.

If:

$$\rho_2 \leqslant \rho_1 e^{-\emptyset_1} \qquad \qquad \text{[Equation 4']}$$

then all effort should be concentrated in A_1. $(\text{i.e. } L_1 = L)$.

If however,

$$\rho_1 > \rho_2 e^{-\emptyset_2} \qquad \text{and} \qquad \rho_2 > \rho_1 e^{-\emptyset_1}$$

[Equation 5] holds. So, by replacing x by L_1, and by use of the
preceding simpler and more meaningful definitions, the optimal
distribution of search effort between the regions may be obtained as
follows:

$$x = \frac{1}{A_1 + A_2} \left(A_1 L + \frac{A_1 A_2}{W} \ln \rho_1 \cdot \frac{1}{\rho_2} \right)$$

Substituting,

$$L_1 = \frac{1}{A}\left(A_1 L + \frac{A_1 A_2}{W} \ln e_1 \frac{1}{e_2}\right)$$

$$= \frac{A_1 L}{A} + \frac{A_1 A_2}{AW}\left(\ln e_1 - \ln e_2\right)$$

Multiplying both sides by $\frac{W}{A_1}$,

$$\frac{WL_1}{A_1} = \frac{WL}{A} + \frac{A_2}{A}\left(\ln e_1 - \ln e_2\right)$$

Noting that $A_2 = A - A_1$ and substituting,

$$\varnothing_1 = \frac{\Phi}{A} + \frac{1}{A}\left[(A - A_1)\ln e_1 - A_2 \ln e_2\right]$$

Finally,

$$\varnothing_1 = \ln e_1 - (A_1 \ln e_1 + A_2 \ln e_2) + \frac{\Phi}{A}$$

[Equation 5']

and similarly,

$$\varnothing_2 = \ln e_2 - (A_1 \ln e_1 + A_2 \ln e_2) + \frac{\Phi}{A}$$

Equations 3', 4', and 5' are the results Koopman obtained in his study of a simple two-region searching situation, and those upon which most further work by Koopman and others in search theory is based.

III. AN n-REGION DISCRETE MODEL

The two-region search model can be fairly readily extended to
a situation involving n regions. Some previous definitions can
be modified as follows:

A_i . . . area of the ith region
L_i . . . length of the observer's track in A_i.
P_i . . . probability that the target is in A_i.

Still assuming that the search is random, the probability of
detection becomes:

$$P(D) = \sum_{i=1}^{n} P_i \left(1 - e^{-\frac{WL_i}{A_i}}\right) = 1 - \sum_{i=1}^{n} P_i e^{-\frac{WL_i}{A_i}}$$

and the n-region problem may be stated as:

maximize
$$P(D) = 1 - \sum_{i=1}^{n} P_i e^{-\frac{WL_i}{A_i}}$$

subject to:
$$\sum_{i=1}^{n} L_i = L$$
$$L_i \geq 0 \quad \text{for} \quad i = 1, 2, \ldots, n$$

This should be compared with the mathematical statement of the
two-region problem.

By an argument analogous to that in the two-region case, it
becomes obvious that either all the available searching effort (Φ)
is divided up among all of the n regions, or there exists some sort
of threshold value for each region below which no effort should be
expended in that region. So, Kuhn-Tucker conditions are applied
to the problem:

minimize
$$\sum_{i=1}^{n} P_i e^{-\frac{WL_i}{A_i}}$$

subject to:
$$\sum_{i=1}^{n} L_i = L$$
$$L_i \geq 0 \quad \text{for} \quad i = 1, 2, \ldots, n$$

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or, letting $x_i = L_i$ as before:

$$\text{minimize} \quad f(x) = \sum_{i=1}^{n} p_i e^{-\frac{Wx_i}{A_i}}$$

$$\text{subject to:} \quad \sum_{i=1}^{n} x_i = X$$

$$x_i \geq 0 \quad \text{for} \quad i = 1, 2, \ldots, n$$

The technique for solving this problem by using the Kuhn-Tucker conditions is set forth in [13], and it is sufficient to the purpose of this paper that only a brief mention be made of the method.

A solution to the minimization problem above is obtained by finding a saddle point of its Lagrangian,

$$L(x, \lambda) = \sum_{i=1}^{n} p_i e^{-\frac{Wx_i}{A_i}} - \lambda \left(X - \sum_{i=1}^{n} x_i \right)$$

In other words, by finding x* and λ* such that,

$$L(x, \lambda^*) \leq L(x^*, \lambda^*) \leq L(x^*, \lambda)$$

Necessary conditions for the existence of such a point were found by Kuhn and Tucker to be:

$$L_{\bar{x}} \geq 0 \quad , \quad \bar{x} L_{\bar{x}} = 0 \quad , \quad \bar{x} \geq 0$$

also,

$$L_{\bar{\lambda}} \leq 0 \quad , \quad \bar{\lambda} L_{\bar{\lambda}} = 0 \quad , \quad \bar{\lambda} \geq 0$$

where

$$\bar{x} = (x_1, x_2, \ldots, x_n), \quad \bar{\lambda} = (\lambda_1, \ldots, \lambda_m), \quad L_{\bar{x}} = \begin{pmatrix} \frac{\partial L}{\partial x_1} \\ \vdots \\ \frac{\partial L}{\partial x_n} \end{pmatrix},$$

and

$$L_{\bar{\lambda}} = \begin{pmatrix} \frac{\partial L}{\partial \lambda_1} \\ \vdots \\ \frac{\partial L}{\partial \lambda_m} \end{pmatrix}.$$

For the problem at hand, with convex objective function and concave constraints, these conditions are also sufficient.

The solution of these conditions for the problem stated above gives the following results for the amount of effort to be expended in any region, i.

Letting $J^* = $ maximum $P(D)$, there exists a dual variable

$$\lambda = \frac{\partial J^*}{\partial x} > 0 \qquad \text{such that:}$$

(a) if $\dfrac{W_{pi}}{A_i} < \lambda$, then $x_i = 0$

(b) if $\dfrac{W_{pi}}{A_i} \geq \lambda$, then $x_i = \dfrac{A_i}{W} \ln \dfrac{W_{pi}}{\lambda A_i}$

Note that λ is the threshold value of $\dfrac{W_{pi}}{A_i}$, which equals W_{ei} , to be more consistent with prior notation, below which no effort is to be expended in region i.

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IV. A CONTINUOUS MODEL

It is a simple step from the n-region discrete model to a continuous one; n will be permitted to increase without bound, hence each A_i approaches zero in the limit. Again, some modification of previous notation is necessary:

$\emptyset_i \equiv \dfrac{WL_i}{A_i}$, similar to the two-region search density.

$A_i \equiv \Delta x \Delta y$, since it represents a two-dimensional area.

As a consequence of the breaking down of the notation into one-dimensional quantities, the subscripts which previously denoted areas are inconsistent and will be replaced by double subscripts, each representing a component in either the x or y direction of the two-dimensional Cartesian coordinate system. Hence, each of the following identities is defined:

$$P_{ij} \equiv P_i$$

$$\emptyset_{ij} \equiv \emptyset_i \qquad \text{, as defined above,}$$

where the single subscript, denoting a region, is not to be confused with the first letter of the double subscript, denoting only the x direction.

The continuous problem, then, is derived as follows:

$$\lim_{\Delta x, \Delta y \to 0} \sum_i \sum_j \frac{P_{ij}}{\Delta x \Delta y} \left(1 - e^{-\emptyset_{ij}}\right) \Delta x \Delta y$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(x,y) \left(1 - e^{-\emptyset(x,y)}\right) dx \, dy$$

where: $P(x,y)$ is the joint probability density, and

$\emptyset(x,y)$ is the density of searching effort.

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A mathematical statement of the continuous problem is:

$$\underset{\phi(x,y)}{\text{maximize}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\left(1 - e^{-\phi(x,y)}\right) dx\, dy$$

$$\text{subject to: } \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x,y)\, dx\, dy = \Phi$$

$$\phi(x,y) \geq 0 \quad \forall x, y$$

This problem was originally treated by Koopman in [5] as a problem in calculus of variations. Since 1946, however, new mathematical methods have evolved which can be used in this case. In a report by J. Taylor [11], Koopman's problem is solved by the Pontriagin Maximum Principle.

Consider, for clarity, the one-dimensional continuous problem:

$$\underset{\phi(x)}{\text{maximize}} \int_{-\infty}^{\infty} p(x)\left(1 - e^{-\phi(x)}\right) dx$$

$$\text{subject to: } \int_{-\infty}^{\infty} \phi(x)\, dx = \Phi$$

$$\phi(x) \geq 0 \quad \forall x$$

Note that the two-dimensional problem which has been considered up to this point, may be treated in the same manner, though the details become complicated, so nothing is lost in considering the latter.

In order to derive a solution to the above problem by the maximum principle, an equivalent problem must be formulated. Defining a new state variable, $y(x)$, equal to the cumulative search effort expended for $X \leq x$:

$$y(x) = \int_{-\infty}^{x} \phi(x)\, dx$$

the problem becomes:

$$\text{maximize} \atop \phi(x) \qquad \int_{-\infty}^{\infty} p(x)\left(1-e^{-\phi(x)}\right) dx$$

$$\text{subject to:} \qquad \frac{dy}{dx} = \phi(x)$$

where $\phi(x) \geq 0$,

and $y(x= -\infty)=0$, $y(x= +\infty)= \Phi$

The solution is, briefly, as follows:

the Hamiltonian is:

$$H\left(x, y(x), \lambda(x), \phi(x)\right) = p(x)\left(1-e^{-\phi(x)}\right) + \lambda \phi(x)$$

and $J*$ is defined as in the n-dimensional case:

$$J^* = \text{maximum} \atop \phi(x) \qquad \int_{-\infty}^{\infty} p(x)\left(1-e^{-\phi(x)}\right) dx$$

Thus the dual variable $\lambda(x) = \frac{\partial J^*}{\partial y(x)} < 0$, since $y(x)$ is the cumulative

search effort, and since by expending more than the optimum effort,

the maximum P(D) can only be decreased. Note that $\lambda(x)$ is a constant

since $\frac{d\lambda(x)}{dt} = -\frac{\partial H}{\partial y} = 0.$

The problem now becomes:

$$\text{maximize} \atop \phi(x) \qquad H(x, y, \lambda, \phi)$$

$$\text{subject to:} \qquad \phi(x) \geq 0$$

which is simplified by letting $\lambda = -u$ where $u>0$.

By applying the necessary and sufficient conditions:

(a) $\phi(x)=0$ when $\frac{\partial H}{\partial \phi} < 0$

(b) $\phi(x)>0$ determined by $\frac{\partial H}{\partial \phi} = p(x)e^{-\phi(x)} - u = 0$

when $\frac{\partial^2 H}{\partial \phi^2} = -p(x) e^{-\phi(x)} < 0 \qquad \forall x, \phi.$

The optimal distribution of searching effort is found to be

dependent upon a threshold value of $p(x)$ called $u>0$ such that:

(1) for $p(x)<u$, $\phi(x)=0$

(2) for $p(x) \geq u$, $\phi(x)= \ln\left(\frac{p(x)}{u}\right).$

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where u is determined as follows:

Defining $\Omega = \left\{ x \mid p(x) \geq u \right\}$, u is chosen such that:

$$\int_{\Omega} \ln\left(\frac{p(x)}{u}\right) dx = \int_{-\infty}^{\infty} \phi(x) dx = \bar{\Phi}$$

The three basic models for the allocation of searching effort have now been described in some detail. The two-dimensional discrete and the continuous models were originally formulated by Koopman in 1946, while the n-dimensional discrete case simply provides an easy passage between the two. As stated previously, numerous extensions of Koopman's basic models have been derived since his original paper, and brief descriptions of some of the more important of these will occupy the remainder of this thesis.

V. SOME EXTENSIONS TO THE BASIC THEORY

A. CHARNES AND COOPER, 1958

Charnes and Cooper state the objective of showing that search theory and mathematical programming can be combined to treat broader classes of operations research problems.

Koopman's discrete allocation problem is treated as a problem in convex programming, the solution of which is obtained by application of the Kuhn-Tucker conditions, and a note is made on the extension of this method to the continuous case.

The Charnes and Cooper model is an important contribution to search theory in that the algorithm is solvable by computer.

B. DE GUENIN, 1961

De Guenin generalized the Koopman models by developing an algorithm in which no assumption is made regarding the detection probability function since, as he points out, while the assumption of the negative exponential (random search) function is adequate for military search problems, there are numerous non-military applications of search theory, such as mine prospecting or oil exploration, where this is invalid.

De Guenin, then, expresses the probability of detection as a function of the density of search effort, that is $p[\phi(x)]$, and the generalized problem becomes:

$$\underset{\phi(x)}{\text{maximize}} \quad P = \int_{-\infty}^{\infty} g(x)\, p[\phi(x)]\, dx = P(\phi)$$

$$\text{subject to:} \quad \int_{-\infty}^{\infty} \phi(x)\, dx = \Phi$$

$$\phi(x) > 0 \quad \forall x$$

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where: P is defined to be the overall probability of success of the search, and g(x) is to be identified with p(x) in Koopman's continuous model. Necessary conditions for the optimal distribution of searching effort are developed by a finite difference approach, although today the problem may be solved by a routine application of the Pontriagin Maximum Principle [11].

C.   DOBBIE, 1963

Dobbie developed sufficient conditions for the additive property of the optimal distribution of search effort noted by Koopman in 1946. That is, the property that the distribution of effort, call it $\emptyset^*(x)$, which maximizes the detection probability with a given amount of effort, Φ , is in fact the sum of the optimal distribution of some part of the effort, E_1, and of the conditionally optimal distribution of the remaining effort, E_2, given that the target has not been found with E_1. The author then derives the optimal distribution working from this property, which is, in fact, nothing more than the Principle of Optimality from Dynamic Programming.

D.   POLLOCK, 1964

Pollock presents a Bayesian approach to the problem of allocating search effort. Decisions are made in a sequential manner, depending on what has been observed up until that time. He has determined the optimal sequential strategies for a two-region discrete model. In Koopman's original report, [5], it is interesting to note that he had shwon nothing was to be gained by such an approach since, in the two-region search, if additional effort, Φ' , is added to Φ after the search has failed to detect the target, and if this Φ is distributed

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optimally between A_1 and A_2, the total effort applied to the two regions is $\emptyset_1 + \emptyset_1'$ and $\emptyset_2 + \emptyset_2'$, respectively, which by use of [Equation 5'] reduce to precisely the same values they would have been had the original effort been $\Phi + \Phi'$ with no prior knowledge of search results.

VI. CONCLUSIONS

The purpose of this thesis has been to bring together under a single cover the classic models for determining the optimal allocation of searching effort, and to give a very brief description of several of the more important recent developments in search theory. An excellent listing of further published works dealing with the allocation of searching effort can be found in reference [3].

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13. ABSTRACT

The problem of optimally allocating available effort to search for an object at sea comprises a major class of problems in naval warfare. This thesis presents in some detail Koopman's classic two-region and continuous search models, along with the n-region discrete model which provides some continuity between the two. Brief summaries of four of the more important extensions to the basic theory are also included.

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